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Ergodic Theory · Exercises · Sheet 7 Discussion on **Thursday, 28 May 2026**

Exercise 7.1 (On lower and upper density). Recall that for a subset $A \subset \mathbb{N}$ the numbers

$$\underline{\mathrm{d}}(A) = \liminf_{N\to\infty} \frac{|\{1,\dots,N\}\cap A|}{N} \quad \text{and} \quad \overline{\mathrm{d}}(A) = \limsup_{N\to\infty} \frac{|\{1,\dots,N\}\cap A|}{N}$$

are the *lower* and *upper density* of A, respectively. Show the following assertions for subsets $A, B \subset \mathbb{N}$.

(a) $0 \le \underline{\mathrm{d}}(A) \le \overline{\mathrm{d}}(A) \le 1$.
(b) $\underline{\mathrm{d}}(\emptyset) = \overline{\mathrm{d}}(\emptyset) = 0$ and $\underline{\mathrm{d}}(\mathbb{N}) = \overline{\mathrm{d}}(\mathbb{N}) = 1$.
(c) If $A \subset B$, then $\underline{\mathrm{d}}(A) \le \underline{\mathrm{d}}(B)$ and $\overline{\mathrm{d}}(A) \le \overline{\mathrm{d}}(B)$.
(d) $\overline{\mathrm{d}}(A) + \underline{\mathrm{d}}(\mathbb{N}\setminus A) = 1$.
(e) $\overline{\mathrm{d}}(A\cup B) \le \overline{\mathrm{d}}(A) + \overline{\mathrm{d}}(B)$.
(f) If $\mathrm{d}(B) = 1$, then $\underline{\mathrm{d}}(A\cap B) = \underline{\mathrm{d}}(A)$ and $\overline{\mathrm{d}}(A\cap B) = \overline{\mathrm{d}}(A)$.
(g) If $\mathrm{d}(B) = 0$, then $\underline{\mathrm{d}}(A\cup B) = \underline{\mathrm{d}}(A)$ and $\overline{\mathrm{d}}(A\cup B) = \overline{\mathrm{d}}(A)$.
(h) For $r, s \in [0,1]$ with $r \le s$ there is a subset $C \subset \mathbb{N}$ with $\underline{\mathrm{d}}(C) = r$ and $\overline{\mathrm{d}}(C) = s$.
Hint: In the case $0 < r \le s < 1$ you can proceed as follows: Set $M_0 \coloneqq 0$. Then, if for some $k \in \mathbb{N}$, the numbers $N_1, \dots, N_{k-1} \in \mathbb{N}$ and $M_0, \dots, M_{k-1} \in \mathbb{N}$ have already been constructed, let $N_k \in \mathbb{N}$ be the smallest $N > M_{k-1}$ with

$$\frac{1}{N}\left(\sum_{j=1}^{k-1}(N_j - M_{j-1}) + (N - M_{k-1})\right) > s,$$

and $M_k \in \mathbb{N}$ be the smallest $M > N_k$ with

$$\frac{1}{M}\sum_{j=1}^{k}(N_j - M_{j-1}) < r.$$

Show that the set $C \coloneqq \bigcup_{j=1}^\infty \{M_{j-1}+1, \dots, N_j\}$ does the trick.

Exercise 7.2 (Syndetic sets). We call a subset $A \subset \mathbb{N}$ *syndetic* (or say that it *has bounded gaps*) if there exists some $N \in \mathbb{N}$ such that $\{n, \dots, n+N\} \cap A \ne \emptyset$ for each $n \in \mathbb{N}$.

(a) Show that if $A \subset \mathbb{N}$ is syndetic, then $\underline{\mathrm{d}}(A) > 0$.
(b) Find an example of a subset $A \subset \mathbb{N}$ which is not syndetic but satisfies $\underline{\mathrm{d}}(A) > 0$.

Exercise 7.3 (Density of the set of primes). Denote by $\mathbb{P}$ the set of all prime numbers. Let further $\pi(N) \coloneqq |\mathbb{P}\cap\{1,\dots,N\}|$ for $N \in \mathbb{N}$. The *prime number theorem* states that $\lim_{N\to\infty} \pi(N)\cdot\frac{\log(N)}{N} = 1$.

(a) Deduce $\mathrm{d}(\mathbb{P}) = 0$ from the prime number theorem.
(b) Show that $n^{\pi(2n)-\pi(n)} \le \binom{2n}{n} \le 4^n$ for each $n \in \mathbb{N}$.
Hint: Notice that for $n \in \mathbb{N}$ each prime number in $\{n+1, \dots, 2n\}$ appears in the prime factorization of $\binom{2n}{n}$.
(c) Use part (b) to show the inequality $\pi(2^{n+1}) - \pi(2^n) \le \frac{2^{n+1}}{n}$ for every $n \in \mathbb{N}$.
(d) Use part (c) to show the inequality $\pi(2^{2n}) - \pi(2) \le 2^{n+1} + \frac{2^{2n+1}}{n}$ for every $n \in \mathbb{N}$.
(e) Use part (d) to deduce $\mathrm{d}(\mathbb{P}) = 0$ without using the prime number theorem.

Exercise 7.4 (Finitary van der Waerden). Show that van der Waerden's theorem from the lectures is equivalent to the following finitary version:

(Finitary van der Waerden) For every $m, k \in \mathbb{N}$ there is $N \in \mathbb{N}$ such that for each finite partition $\{1, \dots, N\} = C_1 \cup \dots \cup C_m$ some set C_j contains an arithmetic progression of length k.

Hint: If the finitary version does not hold, we find some $m, k \in \mathbb{N}$ such that for each $N \in \mathbb{N}$ there is a finite partitition $\{1, \dots, N\} = C_1^N \cup \dots \cup C_m^N$ such that no C_j^N contains a k-term arithmetic progression. Using a digonalization argument, construct a strictly increasing sequence $(N_l)_{l\in\mathbb{N}}$ in $\mathbb{N}$

with the property that for each $n \in \mathbb{N}$ there is some $j \in \{1, \ldots, m\}$ with $n \in C_j^{N_l}$ for all large enough l (in other words: each given number n eventually always has the same color with respect to the N_lth partitions).