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Ergodic Theory · Exercises · Sheet 6 — Discussion on **Thursday, 21 May 2026**

Exercise 6.1 (Weyl's equidistribution criterion). (a) Write μ for the Haar measure on $\mathbb{T}$. Show that for a sequence $(a_n)_{n\in\mathbb{N}}$ in $\mathbb{T}$ the following assertions are equivalent.

(i) The sequence $(a_n)_{n\in\mathbb{N}}$ is equidistributed in $\mathbb{T}$.

(ii) $\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^N f(a_n) = \int_{\mathbb{T}} f \,\mathrm{d}\mu$ for every $f \in \mathrm{C}(\mathbb{T})$.

(iii) $\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^N a_n^m = 0$ for every $m \in \mathbb{Z}\setminus\{0\}$.

Hint: For "(iii) $\Rightarrow$ (ii)" use that the functions $\chi_m\colon \mathbb{T}\to\mathbb{C}$, $z\mapsto z^m$ for $m\in\mathbb{Z}$ span a dense linear subspace of $\mathrm{C}(\mathbb{T})$.

(b) Using (a) show that for $\lambda\in\mathbb{T}$ the sequence $(\lambda^n)_{n\in\mathbb{N}}$ is equidistributed in $\mathbb{T}$ if and only if λ is not a root of unity.

Exercise 6.2 (Wiener's lemma). Show that for each complex Borel measure μ on $\mathbb{T}$ we have

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^N |\hat{\mu}(n)|^2 = \sum_{a\in\mathbb{T}} |\mu(\{a\})|^2.$$

Hint: Rewrite $\frac{1}{N}\sum_{n=1}^N \hat{\mu}(n)\overline{\hat{\mu}(n)}$ for $N\in\mathbb{N}$ suitably by applying Fubini's theorem to the product measure $\mu\times\overline{\mu}$ on $\mathbb{T}\times\mathbb{T}$.

Exercise 6.3 (Weak mixing of powers). Let $k\in\mathbb{N}$. Show that a measure-preserving system (X,μ,T) is weakly mixing if and only if (X,μ,T^k) is weakly mixing.

Exercise 6.4 (On density and weak mixing). Show the following assertions.

(a) If a subset $A\subset\mathbb{N}$ has density $\mathrm{d}(A)$, then $\mathbb{N}\setminus A$ has density $\mathrm{d}(\mathbb{N}\setminus A) = 1-\mathrm{d}(A)$.

(b) If $A,B\subset\mathbb{N}$ have density $\mathrm{d}(A)=\mathrm{d}(B)=0$, then also $\mathrm{d}(A\cup B)=0$.

(c) If $A,B\subset\mathbb{N}$ have density $\mathrm{d}(A)=\mathrm{d}(B)=1$, then also $\mathrm{d}(A\cap B)=1$.

(d) For a weakly mixing measure-preserving system (X,μ,T) there is a strictly increasing sequence $(n_k)_{k\in\mathbb{N}}$ in $\mathbb{N}$ such that

$$\lim_{k\to\infty}\langle T^{n_k}f,g\rangle = \int_X f\,\mathrm{d}\mu\cdot\int_X \overline{g}\,\mathrm{d}\mu \quad \text{for all } f,g\in\mathrm{L}^2(X).$$

Hint: Use that, since the σ-algebra of (X,μ) is countably generated up to nullsets, the spaces $\mathrm{L}^p(X,\mu)$ are separable for $1\le p<\infty$. Combine this fact with (c) to construct a suitable subsequence via the diagonalization technique.