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Exercise 5.1 (On Birkhoff's ergodic theorem)**.** For a measure-preserving system (X, μ, T) show the following assertions.

(a) Birkhoff's individual ergodic theorem implies von Neumann's mean ergodic theorem and for each $f \in \mathrm{L}^2(X, \mu)$ we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} T^n f(x) = Pf(x) \quad \text{for } \mu\text{-almost every } x \in X$$

where P is the mean ergodic projection.

(b) For each $f \in \mathrm{L}^1(X, \mu)$ we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} T^n f(x) = \mathbb{E}(f|\Sigma')(x) \quad \text{for } \mu\text{-almost every } x \in X$$

where Σ' is the σ-algebra of invariant subsets (see Exercise 4.3).

(c) If (X, μ, T) is ergodic and $f \in \mathrm{L}^1(X, \mu)$, then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} T^n f(x) = \int_X f \,\mathrm{d}\mu \quad \text{for } \mu\text{-almost every } x \in X.$$

Exercise 5.2 (Existence of topologically transitive points)**.** Let (X, T) be a topological dynamical system and let μ be an ergodic invariant Borel probability measure on X (cf. Exercise 3.4). Assume further that μ has full support, i.e., $\mu(U) > 0$ for every non-empty open subset $U \subset X$. Show that for μ-almost every $x \in X$ the orbit $\{T^n(x) \mid n \in \mathbb{N}_0\}$ is dense in X.

Exercise 5.3 (More on maximal inequalities)**.** For a probability space (X, μ) let $(T_n)_{n\in\mathbb{N}}$ be a sequence of bounded linear operators on $\mathrm{L}^1(X, \mu)$ with maximal operator T^*, and let

$$F := \{f \in \mathrm{L}^1(X, \mu) \mid \lim_{n\to\infty} T_n f \text{ exists } \mu\text{-almost everywhere}\} \subset \mathrm{L}^1(X, \mu).$$

Consider the following statements.

(i) $F = \mathrm{L}^1(X, \mu)$, i.e., for each $f \in \mathrm{L}^1(X, \mu)$ the limit $\lim_{n\to\infty} T_n f$ exists μ-almost everywhere.
(ii) For each $f \in \mathrm{L}^1(X, \mu)$ we have $T^* f(x) < \infty$ for μ-almost every $x \in X$.
(iii) T^* satisfies a maximal inequality.
(iv) F is closed.

Show that (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii) $\Rightarrow$ (iv). Show further that if F is dense in $\mathrm{L}^1(X, \mu)$, then we even have (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv).

Hint for "(ii) $\Rightarrow$ (iii)": Consider the decreasing function

$$c\colon (0, \infty) \to [0, 1], \quad \lambda \mapsto \sup_{\|f\|\le 1} \mu([T^* f > \lambda]).$$

For fixed $\varepsilon > 0$ use the fact that almost everywhere convergence implies convergence in measure to show that the sets

$$M_{m,n} := \left\{ f \in \mathrm{L}^1(X, \mu) \,\middle|\, \mu\left(\left[\sup_{1\le k\le n} |T_k f| > m\right]\right) \le \varepsilon \right\}$$

are closed in $\mathrm{L}^1(X, \mu)$ for all $m, n \in \mathbb{N}$. Then apply Baire's theorem to find $m \in \mathbb{N}$ and a ball B of radius $\delta > 0$ with $\mu([T^* f > m]) \le \varepsilon$ for each $f \in B$. Finally, use the L^1-subadditivity of T^* to conclude that $c(\frac{2m}{\delta}) \le 2\varepsilon$.

Exercise 5.4 (Characterizations of unique ergodicity)**.** Show that for a topological dynamical system (X,T) the following assertions are equivalent.

(i) (X,T) is uniquely ergodic with (unique) invariant Borel probability measure μ.

(ii) There is an invariant Borel probability measure μ on X such that

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f\circ T^n = \int_X f \,\mathrm{d}\mu \cdot \mathbb{1}$$

in the supremum norm for each $f \in \mathrm{C}(X)$.

(iii) There is an invariant Borel probability measure μ on X such that

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f\circ T^n = \int_X f \,\mathrm{d}\mu \cdot \mathbb{1}$$

pointwise for each $f \in \mathrm{C}(X)$.

(iv) For each $f \in \mathrm{C}(X)$ there is $c_f \in \mathbb{C}$ such that $\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f\circ T^n = c_f \cdot \mathbb{1}$ pointwise.