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Exercise 4.1 (On spectral properties)**.** Show the following assertions for a bounded linear operator $T\colon E \to E$ on a Banach space E.

(a) If T is invertible, then $\sigma(T^{-1}) = \{\lambda^{-1} \mid \lambda \in \sigma(T)\}$.
Hint: For $\lambda \in \mathbb{C} \setminus \{0\}$ *write* $\lambda^{-1} - T^{-1} = (-\lambda^{-1})T^{-1}(\lambda - T)$.

(b) If T is an invertible isometry, then $\sigma(T) \subset \mathbb{T}$.

(c) If T is a non-invertible isometry, then $\sigma(T) = \{\lambda \in \mathbb{C} \mid |\lambda| \leq 1\}$.
Hint: You may use the fact that the boundary of $\sigma(T)$ *is contained in the approximate point spectrum* $\mathrm{A}\sigma(T) \subset \sigma(T)$*, i.e., for every* $\lambda \in \partial\sigma(T)$ *there is a sequence* $(f_n)_{n\in\mathbb{N}}$ *in* E *with* $\|f_n\| = 1$ *for every* $n \in \mathbb{N}$ *and* $\lim_{n\to\infty} \|(\lambda - T)f_n\| = 0$*. Use that* T *is an isometry to deduce* $\mathrm{A}\sigma(T) \subset \mathbb{T}$.

(d) If $T\colon \mathrm{L}^p(\mathbb{T}) \to \mathrm{L}^p(\mathbb{T})$ is the Koopman operator of the rotation system $(\mathbb{T}, a)$ for $a \in \mathbb{T}$ and $p \in [1, \infty)$, then $\mathrm{P}\sigma(T) = \{a^n \mid n \in \mathbb{Z}\}$.

(e) Consider the rotation $(\mathbb{T}^d, a)$ for $a = (a_1, \ldots, a_d) \in \mathbb{T}^d$. Show that the following assertions are equivalent.
  (i) The system $(\mathbb{T}^d, a)$ is ergodic.
  (ii) $a_1, \ldots, a_d$ are $\mathbb{Z}$-independent, i.e., for all $(m_1, \ldots, m_d) \in \mathbb{Z}^d$ the implication
$$a_1^{m_1} \cdots a_d^{m_d} = 1 \quad \Rightarrow \quad m_1 = \cdots = m_d = 0$$
  holds.

Exercise 4.2 (Cesàro convergent sequences)**.** Show the following assertions.

(a) For every sequence of real numbers $(a_n)_{n\in\mathbb{N}}$ the inequalities
$$\liminf_{n\to\infty} a_n \leq \liminf_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} a_n \leq \limsup_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} a_n \leq \limsup_{n\to\infty} a_n$$
hold.

(b) If $(a_n)_{n\in\mathbb{N}}$ is a convergent sequence in $\mathbb{C}$, then $\mathrm{C}\text{-}\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_n$.

(c) The Cesàro limit is shift invariant, i.e., if $(a_n)_{n\in\mathbb{N}}$ is a Cesàro convergent sequence in $\mathbb{C}$, then $(a_{n+1})_{n\in\mathbb{N}}$ is also Cesàro convergent and $\mathrm{C}\text{-}\lim_{n\to\infty} a_n = \mathrm{C}\text{-}\lim_{n\to\infty} a_{n+1}$.

(d) There is a bounded sequence in $\mathbb{C}$ which is not Cesàro convergent.

(e) A sequence $(a_n)_{n\in\mathbb{N}}$ of complex numbers converges if and only if each subsequence of $(a_n)_{n\in\mathbb{N}}$ is Cesàro convergent.

(f) For each $m \in \mathbb{N}$ there is a sequence $(a_n)_{n\in\mathbb{N}}$ of complex numbers such that $\mathrm{C}_{m+1}\text{-}\lim_{n\to\infty} a_n$ exists but $\mathrm{C}_m\text{-}\lim_{n\to\infty} a_n$ does not exist.

Exercise 4.3 (Mean ergodic projections as conditional expectations)**.** Let (X, Σ, μ, T) be a measure-preserving system with mean ergodic projection P. For the subset
$$\Sigma' := \{A \in \Sigma \mid A \text{ is } T\text{-invariant}\} \subset \Sigma$$
show the following assertions.

(a) Σ' is a sub-σ-algebra of Σ.

(b) The conditional expectation[1] of an element $f \in \mathrm{L}^2(X, \Sigma, \mu)$ with respect to Σ' is given by $\mathbb{E}(f|\Sigma') = Pf$.

Exercise 4.4 (Mean ergodic theorem for contractions)**.** Let $T\colon H \to H$ be a bounded linear operator on a Hilbert space H with $\|T\| \leq 1$, and $\mathrm{Fix}(T) = \{f \in H \mid Tf = f\}$ its fixed space.

[1]Recall that for a probability space (X, Σ, μ) and a sub-σ-algebra Σ' the *conditional expectation* of $f \in \mathrm{L}^2(X, \Sigma, \mu)$ with respect to Σ' is the unique element $g \in \mathrm{L}^2(X, \Sigma', \mu|_{\Sigma'}) \subset \mathrm{L}^2(X, \Sigma, \mu)$ such that
$$\int_A f \,\mathrm{d}\mu = \int_A g \,\mathrm{d}\mu \quad \text{for each } A \in \Sigma'.$$
It is denoted by $\mathbb{E}(f|\Sigma')$.

(a) Show that the adjoint operator[2] $T^*\colon H \to H$ satisfies $\mathrm{Fix}(T) = \mathrm{Fix}(T^*)$.
Hint: For $f \in \mathrm{Fix}(T^*)$ *give an upper estimate for* $\|f - Tf\|^2$.

(b) For any bounded linear operator $S\colon H \to H$ one has $\overline{\mathrm{Rg}(S)} = \mathrm{Ker}(S^*)^\perp$. Combine this functional analytic fact[3] with (a) to obtain the orthogonal decomposition $H = \mathrm{Fix}(T) \oplus \overline{\mathrm{Rg}(I - T)}$.

(c) Show that the limit $Pf := \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} T^n f$ exists for each $f \in H$, and that $P\colon H \to H$ is the orthogonal projection onto $\mathrm{Fix}(T)$.

[2]Recall that for any bounded linear operator $T\colon H \to H$ the adjoint T^* is the unique linear operator $S\colon H \to H$ with $\langle Tf, g\rangle = \langle f, Sg\rangle$ for all $f, g \in H$. It is bounded and has the same operator norm as T.

[3]You may also prove it first ;)