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Ergodic Theory · Exercises · Sheet 3 Discussion on **Thursday, 30 April 2026**

Exercise 3.1 (Pointwise almost sure recurrence)**.** Let μ be (the completion of) a Borel probability measure on a compact metric space X. Let further $T\colon X \to X$ be μ-preserving. Show that for μ-almost every $x \in X$ there is a subsequence $(T^{n_k}(x))_{k\in\mathbb{N}}$ of $(T^n(x))_{n\in\mathbb{N}}$ such that $\lim_{k\to\infty} T^{n_k}(x) = x$.
Hint: For each $m \in \mathbb{N}$ cover X with finitely many open balls $B_{m,1}, \dots, B_{m,j_m}$ of radius $\frac{1}{m}$. Then use that each of these balls is infinitely recurrent.

Exercise 3.2 (Hereditary properties of ergodicity)**.** Prove or disprove the following assertions for a measure-preserving system (X, μ, T).

(a) If (X, μ, T) is ergodic, then each factor of (X, μ, T) is ergodic.
(b) If (X, μ, T) is ergodic, then each extension of (X, μ, T) is ergodic.
(c) If (X, μ, T) is ergodic, then also the product system $(X \times X, \mu \times \mu, T \times T)$ is ergodic.
(d) If the product system $(X \times X, T \times T)$ is ergodic, then also (X, μ, T) is ergodic.
(e) If (X, μ, T) is ergodic, then also (X, μ, T^2) is ergodic.
(f) If (X, μ, T^2) is ergodic, then also (X, T) is ergodic.

Exercise 3.3 (Characterizations of ergodicity)**.** Show that for a measure-preserving system (X, μ, T) the following assertions are equivalent[1]:

(i) (X, μ, T) is ergodic.
(ii) Each measurable subset $A \subset X$ with $T^{-1}(A) \subset A$ satisfies $\mu(A) \in \{0, 1\}$.
(iii) Each measurable subset $A \subset X$ with $T^{-1}(A) = A$ satisfies $\mu(A) \in \{0, 1\}$.

Exercise 3.4 (Ergodic measures for topological dynamical systems)**.** Let (X, T) be a topological dynamical system and write $\mathrm{M}_1(X, T)$ for the set of all invariant Borel probability measures of (X, T). An element $\mu \in \mathrm{M}_1(X, T)$ is called *ergodic* if the induced measure-preserving system (X, μ, T) is ergodic.

(a) Let $\mu, \nu \in \mathrm{M}_1(X, T)$ such that μ is ergodic and ν is absolutely continuous[2] with respect to μ. We want to show that then already $\nu = \mu$.
    (1) First make the following auxiliary observation: If (Y, η, S) is any measure-preserving system, then $\eta(A \setminus S^{-1}(A)) = \eta(S^{-1}(A) \setminus A)$ for every measurable subset $A \subseteq Y$.
    Using the Radon–Nikodym theorem we now choose a μ-integrable Borel-measurable function $f\colon X \to [0, \infty)$ with $\int_X f \,\mathrm{d}\mu = 1$ such that $\nu(A) = \int_A f \,\mathrm{d}\mu$ for all Borel subsets $A \subset X$.
    (2) Use (1) for both μ and ν to check that $A := \{x \in X \mid f(x) < 1\}$ is invariant.
    (3) Use (2) to deduce that $\nu = \mu$.

Recall that a subset $C \subset V$ of a complex vector space V is called *convex* if $tx + (1-t)y \in C$ for all $t \in (0, 1)$ and $x, y \in C$. An element $z \in C$ of such a set is called an *extreme point* if the implication

$$z = tx + (1-t)y \text{ for } x, y \in C \text{ and } t \in (0,1) \quad \Rightarrow \quad x = y = z$$

holds.

(b) Show that $\mathrm{M}_1(X, T)$ is a convex subset of the vector space of all complex Borel measures on X.
(c) Show that $\mu \in \mathrm{M}_1(X, T)$ is an extreme point of $\mathrm{M}_1(X, T)$ if any only if μ is ergodic.
    Hint: Use part (a) for the implication "$\Leftarrow$".

[1] The inclusion and equality in (ii) and (iii), respectively, are assumed to hold everywhere (and not up to a nullset).
[2] Recall that for two measures μ and ν on a measurable space X, the measure ν is *absolutely continuous* with respect to μ (in symbols: $\nu \ll \mu$) if $\mu(A) = 0$ for a measurable subset $A \subset X$ always implies $\nu(A) = 0$.