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Exercise 2.1 (On factors and isomorphisms)**.** Let (X,μ) and (Y,ν) be probability spaces.

(a) Let $\varphi\colon X \to Y$ be measure-preserving and $\psi\colon Y \to X$ measurable such that $\psi \circ \varphi = \mathrm{id}_X$ μ-almost everywhere. Show that ψ is measure-preserving.

(b) Let $X' \subseteq X$ be measurable with $\mu(X') = 1$ and $\varphi\colon X' \to Y$ measure-preserving. Show that φ has a measure-preserving extension $\vartheta\colon X \to Y$.

Let now (X,μ,T) be a measure-preserving system.

(c) Show that if $X' \subseteq X$ is measurable with $\mu(X') = 1$, then there is $X'' \subseteq X'$ measurable with $\mu(X'') = 1$ and $T(X'') \subseteq X''$.

Let now also (Y,ν,S) be a measure-preserving system.

(d) Show that the following assertions are equivalent.
  (i) (Y,ν,S) is a factor of (X,μ,T).
  (ii) There is a measurable subset $X' \subseteq X$ with $\mu(X') = 1$ and $T'X' \subseteq X'$ as well as a measure-preserving map $\varphi\colon X' \to Y$ such that $\varphi(Tx) = S(\varphi(x))$ for all $x \in X'$.

(e) Show that the following assertions are equivalent.
  (i) (Y,ν,S) and (X,μ,T) are isomorphic.
  (ii) There are measurable subsets $X' \subseteq X$ and $Y' \subseteq Y$ with $\mu(X') = \nu(Y') = 1$, $TX' \subseteq X'$ and $SY' \subseteq Y'$, as well as a measure-preserving bijection $\varphi\colon X' \to Y'$ with measurable inverse such that $\varphi(Tx) = S(\varphi(x))$ for all $x \in X'$.

Exercise 2.2 (Examples of isomorphisms)**.** (a) For $a \in [0,1)$ consider the measure-preserving system (X,μ,T) defined by $X = [0,1)$ with the Lebesgue measure μ as well as the μ-preserving map $T\colon X \to X$, $x \mapsto x + a \bmod 1$. Moreover, let $b := \mathrm{e}^{2\pi i a} \in \mathbb{T}$ and let (Y,ν,S) be the rotation system on $\mathbb{T}$ defined by b. Show that (X,μ,T) and (Y,ν,S) are isomorphic.

(b) Consider the probability measure $\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$ on $\{0,1\}$ and the corresponding one-sided Bernoulli shift (X,μ,T) from the lectures. Let further (Y,ν,S) be the measure-preserving system defined by the doubling map on $[0,1]$. Show that (X,μ,T) and (Y,ν,S) are isomorphic.
Hint: Consider the maps

$$\begin{aligned}
\varphi\colon \{0,1\}^{\mathbb{N}_0} \to [0,1], &\quad (x_n)_{n\in\mathbb{N}_0} \mapsto \sum_{n=0}^{\infty} \frac{x_n}{2^{n+1}},\\
\psi\colon [0,1] \to \{0,1\}^{\mathbb{N}_0}, &\quad y \mapsto (\lfloor 2^{n+1} y \rfloor \bmod 2)_{n\in\mathbb{N}_0}.
\end{aligned}$$

Exercise 2.3 (Recurrence for topological dynamical systems)**.** A point $x \in X$ of a topological dynamical system (X,T) is called *periodic* if there is $n \in \mathbb{N}$ such that $T^n x = x$.

(a) Give an example of a topological dynamical system (X,T) with $T^n \neq \mathrm{id}_X$ for each $n \in \mathbb{N}$ and a periodic point $x \in X$.

A point $x \in X$ of a topological dynamical system (X,T) is *recurrent* if for each open subset $U \subseteq X$ with $x \in U$ there is some $n \in \mathbb{N}$ such that $T^n x \in U$. Clearly, every periodic point is recurrent.

(b) Give an example of a topological dynamical system (X,T) and a recurrent point $x \in X$ which is not periodic.

(c) Show that each recurrent point $x \in X$ of a topological dynamical system (X,T) is infinitely recurrent, i.e., for each open subset $U \subseteq X$ with $x \in U$ there are infinitely many $n \in \mathbb{N}$ with $T^n x \in U$.

Exercise 2.4 (Topological shift systems)**.** For fixed $k \in \mathbb{N}$ we equip $X \coloneqq \{0, \dots, k-1\}^{\mathbb{N}_0}$ with the metric d defined by

$$d((x_j)_{j\in\mathbb{N}_0}, (y_j)_{j\in\mathbb{N}_0}) \coloneqq \begin{cases} 0 & \text{if } (x_j)_{j\in\mathbb{N}_0} = (y_j)_{j\in\mathbb{N}_0}, \\ \frac{1}{\min\{j\in\mathbb{N}_0 \mid x_j \neq y_j\}+1} & \text{else.} \end{cases}$$

Then X is a compact metric space.

(a) Show that the *left shift* $T\colon X \to X$, $(x_j)_{j\in\mathbb{N}_0} \mapsto (x_{j+1})_{j\in\mathbb{N}_0}$ is continuous and hence defines a topological dynamical system (X, T).

(b) Show that the periodic points of (X, T) form a dense subset of X.