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Exercise 1.1 (Examples of measure-preserving systems). (i) Equip $X = [0, 1]$ with the Lebesgue measure on X. Show that the *tent map*

$$T\colon X \to X, \quad x \mapsto \begin{cases} 2x & \text{if } 0 \leq x < \frac{1}{2}, \\ 2 - 2x & \text{if } \frac{1}{2} \leq x \leq 1, \end{cases}$$

is μ-preserving.

(ii) Equip $X = \mathbb{R}$ with the σ-algebra of Lebesgue measurable subsets and the complete measure

$$\mu\colon \Sigma \to [0, \infty], \quad A \mapsto \int_A \frac{1}{\pi(1 + x^2)} \,\mathrm{d}x.$$

Show that μ is a probability measure and that the *modified Boole transformation*

$$T\colon X \to X, \quad x \mapsto \begin{cases} \frac{1}{2}(x - \frac{1}{x}) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

is μ-preserving.

Hint: Use the substitution rule for $T|_{(0,\infty)}\colon (0, \infty) \to \mathbb{R}$.

Exercise 1.2 (On the Haar measure). Let G be a metrizable compact group and μ its Haar measure. Show the following properties of μ.

(i) $\mu(Aa) = \mu(A)$ for each $a \in G$ and each measurable subset $A \subseteq G$.

(ii) $\mu(A^{-1}) = \mu(A)$ for each measurable subset $A \subseteq G$.

(iii) $\mu(U) > 0$ for each non-empty open subset $U \subseteq G$.
Hint: Construct an open cover of G by translations of U.

(iv) If $T\colon G \to G$ is a continuous and surjective group homomorphism, then T is μ-preserving.

Consider now the case $G = \mathbb{T}$.

(v) Show that for $k \in \mathbb{Z}$ the map $T\colon \mathbb{T} \to \mathbb{T}$, $z \mapsto z^k$ is μ-preserving.

Exercise 1.3 (On torus rotations). (i) Let $a \in \mathbb{T}$ and $k \in \mathbb{N}$ with $a^k = 1$. Describe all invariant (completions of) Borel probability measures for the rotation $(\mathbb{T}, a)$.

(ii) Let $a \in \mathbb{T}$ with $a^k \neq 1$ for all $k \in \mathbb{N}$.

(a) Show that $\{a^k \mid k \in \mathbb{N}\}$ is dense in $\mathbb{T}$.
Hint: First show that for $\varepsilon > 0$ that there is $k \in \mathbb{N}$ with $|a^k - 1| \leq \varepsilon$.

(b) Show that the Haar measure of $\mathbb{T}$ is the only invariant completed Borel measure for the rotation $(\mathbb{T}, a)$.

(iii) Show that

$$\overline{\{\sin(k) \mid k \in \mathbb{N}\}} = [-1, 1] = \overline{\{\cos(k) \mid k \in \mathbb{N}\}}.$$